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A CLASS OF MOTIONS OF A TOP IN THE GORYACHEV-CHAPLYGIN CASE*

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A solution of the Euler-Poisson equations is studied for the Goryachev-Chaplygin case /1/ under the condition when the ultraelliptic integrals degenerate to elliptic /2/. A solution is constructed for a class of motions in which both quantities, u and v, brought in by Chaplygin, vary with time, but one of them tends asymptotically to a constant when the time increases without limit. The dependence of the Euler-Poisson variables on time is expressed in terms of elliptic functions and an elliptic integral of the third kind. Fairly simple approximate formulas are given for determining all six variables sought.

1. Equations of motion. We will use the Goryachev-Chaplygin conditions and take the Euler-Poisson equations in traditional form /1/ (a dot denotes differentiation with respect to time)

$$4p^{\cdot} = 3qr,\ 4q^{\cdot} = -3rp - a\gamma'',\ r^{\cdot} = a\gamma' \qquad (1.1)$$
$$\gamma^{\cdot} = r\gamma' - q\gamma'',\ \gamma'^{\cdot} = p\gamma'' - r\gamma,\ \gamma''^{\cdot} = q\gamma - p\gamma'$$

If the area constant is zero, the above system admits of four algebraic integrals

$$4(p^2 + q^2) + r^2 - 2a\gamma = k,\ \gamma^2 + \gamma'^2 + \gamma''^2 = 1 \qquad (1.2)$$
$$4(p\gamma + q\gamma') + r\gamma'' = 0,\ r(p^2 + q^2) + ap\gamma'' = g$$

We introduce two auxiliary variables u, v so that

$$u + v = r,\ uv = -4(p^2 + q^2) \qquad (1.3)$$

The following differential equations describe how these quantities vary with time:

$$2(u - v)u^{\cdot} = \sqrt{F(u)},\quad 2(v - u)v^{\cdot} = \sqrt{F(v)} \qquad (1.4)$$
$$F(u) = f_1(u) f_2(u)$$
$$f_1(u) = -u^3 + (k + 2a)u + 4g,\ f_2(u) = u^3 + (2a - k)u - 4g$$

System (1.4) can be written in terms of total differentials thus

$$\frac{du}{\sqrt{F(u)}} + \frac{dv}{\sqrt{F(v)}} = 0,\quad \frac{2u\,du}{\sqrt{F(u)}} + \frac{2v\,dv}{\sqrt{F(v)}} = dt \qquad (1.5)$$

Let us write $(k - 2a)^3 + 27\cdot 4g^2 = 0$, or in parametric form (b is an auxiliary constant)

$$k - 2a = 3b^2,\ 2g = -b^3 \qquad (1.6)$$

Then $f_2(u) = (u - b)^2(u + 2b)$, $f_1(u) = -(u - \alpha_1)(u - \alpha_2)(u - \alpha_3)$ where all three roots $\alpha_1, \alpha_2, \alpha_3$ are real and $\alpha_1 < \alpha_2 < 0 < \alpha_3$, $-2b < \alpha_3$. It follows that the polynomial $F(u)$ has a multiple root

$$F(u) = (u - b)^2 R(u),\ R(u) = (u + 2b) f_1(u) \qquad (1.7)$$

Let us describe the type of set in which the variables u, v vary. We shall assume that $g \neq 0$, since when $g = 0$ the solution is known /3, 4/. Then $p^2 + q^2 \neq 0$ and hence by virtue of (1.3), $uv \neq 0$. We can assume without loss of generality that $u > 0, v < 0, b < 0$. Bearing this in mind, we obtain

$$0 < -2b \leqslant u \leqslant \alpha_3 \qquad (1.8)$$

Thus the quantity u varies on the interval (1.8). The set of variations is more complicated for the second variable v. Depending on the initial data, three versions are possible 1) $\alpha_1 \leqslant v < b < 0$, 2) $b < v \leqslant \alpha_2$, 3) $v = b = \text{const}$. The last version is relatively simple and

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its solution is already known /2/

$$4p = (2b - r)\gamma^{\cdot},\ 4(p^2 + q^2) = b(b - r),\ b(r + b) = -a\gamma^{\cdot 2}$$
$$2a(\gamma + 1) = (r + b)(r - 2b),\ b\gamma' = 2q\gamma^{\cdot}$$
$$4(r^{\cdot})^2 = (r + b)[4a(r - b) - (r + b)(r - 2b)^2] \tag{1.9}$$

In the general case when the quantity v is variable, we know /2/ that the motion approaches its limit mode asymptotically, and the mode is determined by (1.9).

2. Constructing the solution.

Using conditions (1.6) and relation (1.7), we can reduce the equations (1.4), (1.5) to the form

$$2(u - v)u^{\cdot} = (u - b)\sqrt{R(u)},\quad 2(v - u)v^{\cdot} = (v - b)\sqrt{R(v)} \tag{2.1}$$

$$\frac{du}{\sqrt{R(u)}} + \frac{dv}{\sqrt{R(v)}} = \frac{dt}{2},\quad \frac{du}{(u - b)\sqrt{R(u)}} + \frac{dv}{(v - b)\sqrt{R(v)}} = 0 \tag{2.2}$$

Let us carry out the bilinear transformation of the variables u, v

$$z_1 = m + \frac{n}{u + 2b},\quad z_2 = m + \frac{n}{v + 2b} \tag{2.3}$$

where m, n are constants, $24m = R''(-2b),\ 4n = R'(-2b)$, and the prime denotes a derivative. In the new variables Eqs.(2.2) take the form (τ_{10}, τ_{20} are arbitrary constants)

$$\frac{dz_1}{\sqrt{\varphi(z_1)}} + \frac{dz_2}{\sqrt{\varphi(z_2)}} = d\tau_1,\quad \frac{dz_1}{(z_1 - b_1)\sqrt{\varphi(z_1)}} + \frac{dz_2}{(z_2 - b_1)\sqrt{\varphi(z_2)}} = -d\tau_2 \tag{2.4}$$
$$\varphi(z) = 4z^3 - g_2 z - g_3,\quad \tau_1 = \frac{t}{2} + \tau_{10},\quad \tau_2 = \frac{t}{2a_1} + \tau_{20}$$

Carrying out all elementary algebra, we obtain

$$a_1 = \frac{2a}{3},\quad b_1 = \frac{2a - 3k}{12},\quad m = a_1 + b_1,\quad n = -3ba_1,\quad g_2 = 12b_1^2,\quad g_3 = -8b_1^3 - 2a_1^3$$

Systems of this type also arises in the study of the Kowalewska case /5–7/. Let us construct the solution of (2.4) in a form suitable for applications. We introduce new variables u_1, u_2 and a constant α such that

$$z_1 = P(u_1),\ z_2 = P(u_2),\ b_1 = P(\alpha) \tag{2.5}$$

where $P(u)$ is an elliptic Weierstrass function with invariants g_2, g_3. Integrating the system obtained from (2.4) we carry out the substitution (2.5) and obtain

$$u_1 + u_2 = \tau_1,\quad I(u_1) + I(u_2) = P'(\alpha)\tau_2,\quad I(u) = P'(\alpha)\int_0^u \frac{du}{P(u) - P(\alpha)} \tag{2.6}$$

This gives us the formal solution. The final relations (2.6) determine the total solution of the initial system (2.1) in implicit form. However, we can write the variables u, v in explicite form, where their final form is represented by simple analytic functions of time. With this aim we shall use, for the integrals $I(u)$, the theorem of addition of argůments /8/ written in the form

$$\frac{\sigma(u_1 - \alpha)\sigma(u_2 - \alpha)\sigma(u_1 + u_2 + \alpha)}{\sigma(u_1 + \alpha)\sigma(u_2 + \alpha)\sigma(u_1 + u_2 - \alpha)} = e^{\varphi} \tag{2.7}$$
$$\varphi = I(u_1) + I(u_2) - I(u_1 + u_2)$$

where $\sigma(u)$ is the Weierstrass entire function.

Using relations (2.6) we can obtain the quantity φ as an explicit function of time

$$\varphi = P'(\alpha)\tau_2 - I(\tau_1) \tag{2.8}$$

Further, the left part of (2.7) can be written in the form

$$\frac{s_\nu(u_1 + u_2 + \alpha)s_\nu(\alpha) + s_\nu(u_1)s_\nu(u_2)}{s_\nu(u_1 + u_2 - \alpha)s_\nu(\alpha) - s_\nu(u_1)s_\nu(u_2)}$$

where $s_\nu(u) = \sigma_\nu(u)/\sigma(u)$ $(\nu = 1, 2, 3)$ are elliptic functions and equation (2.7) is solved for the product $s_\nu(u_1)s_\nu(u_2)$. This yields

$$s_\nu(u_1)s_\nu(u_2) = L_\nu(t) \tag{2.9}$$
$$L_\nu(t) = s_\nu(\alpha)\frac{s_\nu(\tau_1 - \alpha)e^{\varphi} - s_\nu(\tau_1 + \alpha)}{1 + e^{\varphi}},\quad \nu = 1, 2, 3$$

The elliptic functions $s_\nu(u)$ $(\nu = 1, 2, 3)$ are related to the Weierstrass function $P(u)$ very simply: $P(u) = s_1^2(u) + e_1 = s_2^2(u) + e_2 = s_3^2(u) + e_3$ where e_1, e_2, e_3 are roots of the polynomial $\varphi(z) = 4z^3 - g_2 z - g_3 = 4(z - e_1)(z - e_2)(z - e_3)$. Taking into account the relations (2.5) we obtain $s_\nu(u_1) = \sqrt{z_1 - e_\nu}$, $s_\nu(u_2) = \sqrt{z_2 - e_\nu}$, therefore we can write Eqs.(2.9) in the form

$$\sqrt{(z_1 - e_\nu)(z_2 - e_\nu)} = L_\nu(t) \tag{2.10}$$

Relations (2.10) give the required answer. Using these formulas we can find the symmetric functions $z_1 + z_2, z_1 z_2$ as linear combinations of known functions of time $L_\nu(t)$, i.e. as unique analytic functions of time, and then construct the quantities z_1, z_2 themselves.

3. Computation of the Euler-Poisson variables. Having obtained z_1, z_2 from formulas (2.3), we find the Chaplygin u, v variables. Using the four algebraic integrals (1.2) and relations (1.3), we can find the Euler-Poisson variables $p, q, r, \gamma, \gamma', \gamma''$. The final formulas however are very cumbersome.

Let us turn our attention to the asymptotic form of the motion. We can prove it easily now. Indeed, differentiating (2.8) we obtain

$$\frac{d\varphi}{dt} = \frac{P'(\alpha)}{2a_1} \frac{P(\tau_1) + m}{P_1(\tau_1) - b_1}$$

It can be confirmed that $d\varphi/dt > \omega > 0$ where ω is a constant, and in view of this the quantity φ is a monotonic function of time. As $t \to \infty$, we shall have $e^{-|\varphi|} \to 0$, $L_\nu(t) \to s_\nu(\tau_1 \pm \alpha)$ where the sign preceding the constant α depends on the sign of φ. The indicated property of the motion follows from this.

In this connection we can write the variables sought in the form of power series in terms of the small variable quantity $e^{-|\varphi|}$ with periodic coefficients.

Let us construct an approximate expressions for the Euler-Poisson variables, using for simplicity the equations of motion (1.1) directly. The limit motion is realized under an additional condition, provided that $v(0) = b$ at the initial instant. Let us expand the solution in series in powers of the small parameter $\mu = v(0)/b - 1$

$$p = p_0 + \mu p_1 + \ldots, \quad q = q_0 + \mu q_1 + \ldots, \quad r = r_0 + \mu r_1 + \ldots$$
$$a\gamma = \alpha_0 + \mu\alpha_1 + \ldots, \quad a\gamma' = \beta_0 + \mu\beta_1 + \ldots, \quad a\gamma'' = \gamma_0 + \mu\gamma_1 + \ldots \tag{3.1}$$

The principal terms $p_0, q_0, \ldots, \gamma_0$ of these series characterize the limit motion, which is described, in accordance with (1.9), by the relations

$$4ap_0 = (2b - r_0)\gamma_0, \quad 4(p_0^2 + q_0^2) = b(b - r_0), \quad ab(r_0 + b) = \gamma_0^2$$
$$2(\alpha_0 + a) = (r_0 + b)(r_0 - 2b), \quad b\beta_0 = 2q_0\gamma_0, \quad \gamma_0^{\cdot} = -aq_0 \tag{3.2}$$

Substituting series (3.1) into (1.1), (1.2) and comparing terms of the first-order of smallness, we obtain

$$4p_1^{\cdot} = 3(q_0 r_1 + q_1 r_0), \quad 4q_1^{\cdot} = -3(r_0 p_1 + r_1 p_0) - \gamma_1, \quad r_2^{\cdot} = \beta_1 \tag{3.3}$$
$$\alpha_1^{\cdot} = \beta_0 r_1 + \beta_1 r_0 - \gamma_0 q_1 - \gamma_1 q_0, \quad \beta_1^{\cdot} = \gamma_0 p_1 + \gamma_1 p_0^{\cdot} - \alpha_0 r_1 - \alpha_1 r_0^{\cdot}$$
$$\gamma_1^{\cdot} = \alpha_0 q_1 + \alpha_1 q_0 - \beta_0 p_1 - \beta_1 p_0$$
$$4(p_0 p_1 + q_0 q_1) + r_0 r_1 - \alpha_1 = 0, \quad \alpha_0\alpha_1 + \beta_0\beta_1 + \gamma_0\gamma_1 = 0$$
$$4(p_0\alpha_1 + p_1\alpha_0 + q_0\beta_1 + q_1\beta_0) + r_0\gamma_1 + r_1\gamma_0 = 0$$
$$2r_0(p_0 p_1 + q_0 q_1) + r_1(p_0^2 + q_0^2) + \gamma_0 p_1 + \gamma_1 p_0 = 0$$

Let us first find the Chaplygin variables

$$u = u_0 + \mu u_1 + \ldots, \quad v = v_0 + \mu v_1 + \ldots \tag{3.4}$$

Substituting series (3.4) into (2.1) and equating first the principal terms, and then terms of the second order of smallness, we obtain

$$2u_0^{\cdot} = \sqrt{R(u_0)}, \quad v_0^{\cdot} = 0 \tag{3.5}$$

$$(u_0 - v_0)u_1^{\cdot} + (u_1 - v_1)u_0^{\cdot} = f'(u_0)u_1 \tag{3.6}$$
$$(v_0 - u_0)v_1^{\cdot} + (v_1 - u_1)v_0^{\cdot} = f'(v_0)v_1, \quad 2f((u) = (u - b)\sqrt{R(u)})$$

From (3.5) it follows that $v_0 = \text{const} = b$ and the variables u_0 is a periodic function of time.

Let us now turn to system (3.6). The second equation of this system yields $v_1 = C_0 e^{\lambda}$ where C_0 is an arbitrary constant, $\lambda^{\cdot} = l/(b - u_0)$, $l = f'(b) = \frac{1}{2}\sqrt{R(b)} = b\sqrt{3a}$. It can be shown that $\lambda^{\cdot} = \varphi^{\cdot}$ and hence $v_1 = lC_1 e^{\varphi}$ where C_1 is an arbitrary constant. Knowing the dependence of v on time we obtain, from the first equation of (3.6),

$$u_1 = u_0^{\cdot}(C_2 - C_1 e^{\varphi}) \tag{3.7}$$

where C_2 is the second integration constant.

Thus, having obtained the quantities u_0, v_0, u_1, v_1 we can use the relations (1.3), (3.3) to compute the variables $p_1, q_1, r_1, \alpha_1, \beta_1, \gamma_1$. This yields

$$p_1 = p_0^{\cdot}N + \frac{C_1 l}{2ab}(ap_0 + lq_0)e^{\varphi}, \quad q_1 = q_0^{\cdot}N + \frac{C_1 l}{2ab}(-lp_0 + aq_0)e^{\varphi} \tag{3.8}$$
$$r_1 = r_0^{\cdot}N + C_1 l e^{\varphi}, \quad \alpha_1 = \alpha_0^{\cdot}N + \frac{C_1 l}{2}(u_0 + 2b)e^{\varphi}, \quad \beta_1 = \beta_0^{\cdot}N + \frac{(l - \beta_0)l}{b - u_0}C_1 e^{\varphi}$$
$$\gamma_1 = \gamma_0^{\cdot}N + C_1 l\left[\frac{\gamma_0\alpha_0}{2ab} - \frac{2q_0(l - \beta_0)}{b(b - u_0)}\right]e^{\varphi}, \quad N = C_2 - C_1 e^{\varphi}$$

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ON A MODIFICATION OF THE AVERAGING METHOD FOR SEEKING HIGHER APPROXIMATIONS*

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Systems in the N.N. Bogolyubov standard form as well as systems with rapid phases are considered. It is proposed to seek the solution in the form of an asymptotic series in a small parameter with coefficients representable in the form of the sum of two functions. The first depends on slow time and is found as the solution of a simpler equation in a finite segment. The second is a trigonometric polynomial of the time (or the angular displacements) with coefficients which depend on the slow time (it is found in an explicit manner). It is convenient to use the results in solving certain problems in celestial mechanics.

Utilization of the Bogolyubov-Mitropol'skii-Velosov averaging method /1, 2/ in calculating high approximations of a solution with fixed initial condition can be made complicated because of the awkwardness of appropriate manipulations. A modification is proposed below for the method which is based on ideas utilized in the theory of singularly perturbed equations /3, 4/.

Let R^n be an n-dimensional Euclidean space, and let D be a bounded domain in R^n. We assume that a function $X(t, x)$ with values in R^n, all of whose derivatives with respect to x to the $(N+1)$-th order are continuous, is defined in $[0, \infty) \times D$. Let $X(t, x)$ be a trigonometric polynomial in t.

The Cauchy problem

$$dx/dt = \varepsilon X(t, x),\ x(0) = \alpha \in D,\ t \in [0, T/\varepsilon] \tag{1}$$

is considered, where ε is a small positive parameter. We will seek an approximate solution of this problem in the form

$$\begin{gathered} x_* = x_0 + \varepsilon x_1 + \ldots + \varepsilon^N x_N \\ x_i = u_i(\xi) + v_i(\xi, t),\ i = 0, 1, \ldots, N,\ \xi = \varepsilon t \end{gathered} \tag{2}$$

Here v_i are trigonometric polynomials in t. Formally substituting (2) into (1), we have

$$\left[\varepsilon \frac{du_0}{d\xi} + \varepsilon \frac{\partial v_0}{\partial \xi} + \frac{\partial v_0}{\partial t}\right] + \varepsilon \left[\varepsilon \frac{du_1}{d\xi} + \varepsilon \frac{\partial v_1}{\partial \xi} + \frac{\partial v_1}{\partial t}\right] + \ldots = \varepsilon X(t, x_0 + \varepsilon x_1 + \ldots) \tag{3}$$

We shall try to satisfy this equation for all $\xi \in [0, T]$ and $t \in [0, \infty)$. We set $v_0 \equiv 0$. We shall later denote the mean value of the function X with respect to t by $\bar{X}$. Then $X = \bar{X} + X'$. Evidently X' has a zero mean in t. Furthermore

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